

Heat diffusion with frozen boundary

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Abstract

Consider “Frozen Random Walk” on \mathbb{Z} : n particles start at the origin. At any discrete time, the leftmost and rightmost $\lfloor \frac{n}{4} \rfloor$ particles are “frozen” and do not move. The rest of the particles in the “bulk” independently jump to the left and right uniformly. The goal of this note is to understand the limit of this process under scaling of mass and time. To this end we study the following deterministic mass splitting process: start with mass 1 at the origin. At each step the extreme quarter mass on each side is “frozen”. The remaining “free” mass in the center evolves according to the discrete heat equation. We establish diffusive behavior of this mass evolution and identify the scaling limit under the assumption of its existence. It is natural to expect the limit to be a truncated Gaussian. A naive guess for the truncation point might be the 1/4 quantile points on either side of the origin. We show that this is not the case and it is in fact determined by the evolution of the second moment of the mass distribution.

1 Introduction

The goal of this note is to understand the long term behavior of the mass evolution process which is a divisible version of the particle system “Frozen Random Walk”. We define **Frozen-Boundary Diffusion** with parameter α (or **FBD**- α) as follows. Informally it is a sequence μ_t of symmetric probability distributions on \mathbb{Z} . The sequence has the following recursive definition: given μ_t , the leftmost and rightmost $\frac{\alpha}{2}$ masses are constrained to not move, and the remaining $1 - \alpha$ mass diffuses according to one step of the discrete heat equation to yield μ_{t+1} . In other words, we split the mass at site x equally to its two neighbors. Formal descriptions appear later. We briefly remark that this process is similar to Stefan type problems, which have been studied for example in [2].

Now we also introduce the random counterpart of **FBD**- α . We define the frozen random walk process (Frozen Random Walk- $(n, 1/2)$) as follows: n particles start at the origin. At

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any discrete time the leftmost and rightmost $\lfloor n^{\frac{\alpha}{2}} \rfloor$ particles are “frozen” and do not move. The remaining $n - 2\lfloor n^{\frac{\alpha}{2}} \rfloor$ particles independently jump to the left and right uniformly. Letting $n \rightarrow \infty$ and fixing t , the mass distribution for the above random process converges to the t^{th} element, μ_t , in **FBD**- α . However, if t and n simultaneously go to ∞ , one has to control the fluctuations to be able to prove any limiting statement. Figure 1 depicts the mass distribution μ_t and the frozen random walk process for $\alpha = \frac{1}{2}$.

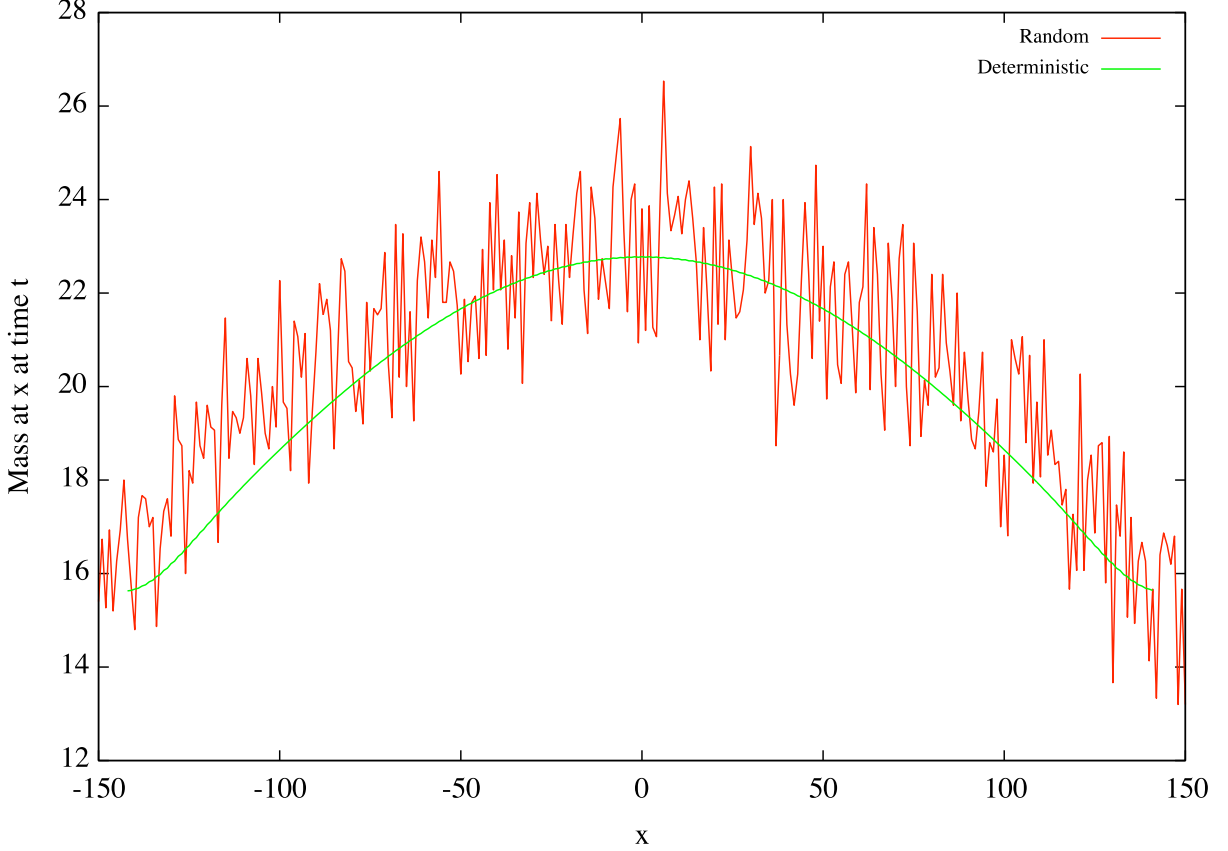


Figure 1: **Frozen-Boundary Diffusion**- $\frac{1}{2}$ and **Frozen Random Walk**-(10000, $\frac{1}{2}$) averaged over 15 trials at $t = 25000$.

At every step t of **FBD**- α , we also keep track of the location of the boundary of the process, β_t , which we define as

$$\beta_t := \sup \left\{ x \in \mathbb{Z} : \mu_t([x, \infty)) \geq \frac{\alpha}{2} \right\}.$$

We will show that

Lemma 1. *For every $\alpha \in (0, 1)$ there exist constants $a, b > 0$ such that*

$$a\sqrt{t} < \beta_t < b\sqrt{t}, \forall t.$$

The lemma above suggests that a proper scaling of β_t is \sqrt{t} . Motivated by this behavior of the boundary β_t , one can ask the following natural questions:

Q1. Does $\frac{\beta_t}{\sqrt{t}}$ converge?

Considering μ_t as a measure on \mathbb{R} , for $t = 0, 1, \dots$ define the Borel measure $\tilde{\mu}_t(\alpha) = \tilde{\mu}_t$ on \mathbb{R} equipped with the Borel σ -algebra such that for any Borel set A ,

$$\tilde{\mu}_t(A) = \mu_t(\{y\sqrt{t} : y \in A\}). \quad (1)$$

We can now ask

Q2. Does the sequence of probability measures $\tilde{\mu}_t$ have a weak limit?

Q3. If $\tilde{\mu}_t$ has a weak limit, what is this limiting distribution?

We conjecture affirmative answers to **Q1** and **Q2**:

Conjecture 1. For every $\alpha \in (0, 1)$, there exists $\ell_\alpha > 0$ such that

$$\lim_{t \rightarrow \infty} \frac{\beta_t}{\sqrt{t}} = \ell_\alpha.$$

Conjecture 2. Fix $\alpha \in (0, 1)$. Then there exists a probability measure $\mu_\infty(\alpha)$ on \mathbb{R} such that as $t \rightarrow \infty$,

$$\tilde{\mu}_t \xrightarrow{\text{weak}} \mu_\infty(\alpha),$$

where $\xrightarrow{\text{weak}}$ denotes weak convergence in the space of finite measures on \mathbb{R} .

That Conj. 2 implies Conj. 1 is the content of Lemma 4. We now state our main result which shows that Conj. 1 implies Conj. 2 and identifies the limiting distribution, thus answering **Q3**. To this end we need the following definition.

Definition 1. Let $\Phi(\cdot)$ be the standard Gaussian measure on \mathbb{R} . Also for any $q > 0$ denote by $\Phi_q(\cdot)$, the probability measure on \mathbb{R} which is supported on $[-q, q]$ and whose density is the standard Gaussian density restricted on the interval $[-q, q]$ and properly normalized to have integral 1.

Theorem 2. Assuming that $\lim_{t \rightarrow \infty} \frac{\beta_t}{\sqrt{t}}$ is a constant, the following is true:

$$\tilde{\mu}_t \xrightarrow{\text{weak}} \mu_\infty(\alpha),$$

where,

$$\mu_\infty(\alpha) = \frac{\alpha}{2} \delta(-q_\alpha) + (1 - \alpha) \Phi_{q_\alpha} + \frac{\alpha}{2} \delta(q_\alpha),$$

and q_α is the unique positive number such that:

$$\frac{\alpha}{2} q_\alpha = \frac{(1 - \alpha) e^{-q_\alpha^2/2}}{\sqrt{2\pi} \Phi([-q_\alpha, q_\alpha])}.$$

Remark 1. It is easy to show (see Lemma 4) that the above result implies that

$$\lim_{t \rightarrow \infty} \frac{\beta_t}{\sqrt{t}} = q_\alpha.$$

Thus observe that by the above result, just assuming that the boundary location properly scaled converges to a constant determines the value of the constant. This is a consequence of uniqueness of the root of a certain functional equation discussed in detail in Section 2.

1.1 Formal definitions

Let $\mathbf{FBD}\text{-}\alpha := \{\mu_0, \mu_1, \dots\}$: where for each $t = 0, 1, \dots$, μ_t is a probability distribution on \mathbb{Z} . For brevity we suppress the dependence on α in the notation since there is no scope of confusion as α will remain fixed throughout any argument.

Let $\mu_0 \equiv \delta(0)$ be the delta function at 0. By construction μ_t will be symmetric for all t . As described in Section 1 each μ_t contains a “constrained/frozen” part and a “free” part. Let the free mass and the frozen mass be denoted by the mass distributions ν_t and f_t respectively.

Recall the boundary of the process,

$$\beta_t = \sup \left\{ x \in \mathbb{Z} : \mu_t([x, \infty)) \geq \frac{\alpha}{2} \right\}. \quad (2)$$

Then for all $y \geq 0$,

$$f_t(y) := \begin{cases} \mu_t(y) & y > \beta_t \\ \frac{\alpha}{2} - \sum_{z > \beta_t} \mu_t(z) & y = \beta_t \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

For $y < 0$ let $f_t(y) := f_t(-y)$. Thus f_t is the extreme $\alpha/2$ mass on both sides of the origin. Define the free mass to be $\nu_t := \mu_t - f_t$. With the above notation the heat diffusion is described by

$$\mu_{t+1}(x) = \frac{\nu_t(x-1) + \nu_t(x+1)}{2} + f_t(x). \quad (4)$$

Recall Lemma 1, which implies the diffusive nature of the boundary:

Lemma 1. For every $\alpha \in (0, 1)$ there exist constants $a, b > 0$ such that

$$a\sqrt{t} < \beta_t < b\sqrt{t}, \forall t.$$

This result implies that in order to obtain any limiting statement about the measures μ_t , one has to scale space down by \sqrt{t} .

The proof of the lemma appears later. Let us first prove that the frozen mass f_t cannot be supported over many points.

Lemma 3. For all t , the frozen mass at time t , f_t , is supported on at most two points on each side of the origin, i.e., for all $y \in \mathbb{Z}$ such that $|y| \geq \beta_t + 2$, we have $f_t(y) = 0$.

Proof. The lemma follows by induction. Assume for all $k \leq t$, for all y such that $|y| \geq \beta_k + 2$, we have $\mu_k(y) = f_k(y) = 0$. The base case $t = 0$ is easy to check. Now observe that by (4) and the above induction hypothesis,

$$\mu_{t+1}(y) = 0, \quad (5)$$

for all $|y| \geq \beta_t + 2$. Also notice that by (4) it easily follows that β_t is a non-decreasing function of t . Thus clearly for all y , with $|y| \geq \beta_{t+1} + 2 \geq \beta_t + 2$,

$$\mu_{t+1}(y) = 0.$$

Hence we are done by induction. □

We now return to the proof of the diffusive nature of the boundary of the process β_t .

Proof of Lemma 1. We consider the second moment of the mass distribution μ_t , which we denote as $M_2(t) := \sum_{x \in \mathbb{Z}} \mu_t(x)x^2$. This is at most $(\beta_t + 1)^2$ since μ_t is supported on $[-\beta_t - 1, \beta_t + 1]$ by Lemma 3. It is also at least $\alpha\beta_t^2$ since there exists mass α which is at a distance at least β_t from the origin. Now we observe how the second moment of the mass distribution evolves over time. Suppose a free mass m at x splits and moves to $x - 1$ and $x + 1$. Then the increase in the second moment is

$$\frac{m}{2}((x+1)^2 + (x-1)^2) - mx^2 = m.$$

Since at every time step exactly $1 - \alpha$ mass is moving, the net change in the second moment at every step is $1 - \alpha$. So at time t the second moment is exactly

$$t(1 - \alpha). \tag{6}$$

Hence $\alpha\beta_t^2 < t(1 - \alpha) < (\beta_t + 1)^2$, and we are done. \square

We next prove Conjecture 1 (a stronger version of Lemma 1) assuming Conjecture 2.

Lemma 4. *If Conjecture 2 holds, then so does Conjecture 1, i.e., for every $\alpha \in (0, 1)$, there exists $\ell_\alpha > 0$, such that*

$$\lim_{t \rightarrow \infty} \frac{\beta_t}{\sqrt{t}} = \ell_\alpha.$$

Proof. Fix $\alpha \in (0, 1)$. From Lemma 1 we know that $\{\beta_t/\sqrt{t}\}$ is bounded. Hence, if β_t/\sqrt{t} does not converge, there exists two subsequences $\{s_1, s_2, \dots\}$ and $\{t_1, t_2, \dots\}$ such that

$$\lim_{i \rightarrow \infty} \beta_{s_i}/\sqrt{s_i} = q_1 \text{ and } \lim_{j \rightarrow \infty} \beta_{t_j}/\sqrt{t_j} \rightarrow q_2,$$

for some $q_2, q_1 > 0$ such that $q_2 - q_1 := \delta > 0$. Recall $\mu_\infty(\alpha)$ from Conjecture 2. Now by hypothesis,

$$\lim_{i \rightarrow \infty} \tilde{\mu}_{s_i} \xrightarrow{\text{weak}} \mu_\infty(\alpha), \quad \lim_{j \rightarrow \infty} \tilde{\mu}_{t_j} \xrightarrow{\text{weak}} \mu_\infty(\alpha).$$

This yields a contradiction since the first relation implies $\mu_\infty(\alpha)$ assigns mass 0 to the interval $(q_2 - \frac{\delta}{2}, q_2 + \frac{\delta}{2})$ while the second one implies (by Lemma 3) that it assigns mass at least $\frac{\alpha}{2}$ to that interval. \square

2 Proof of Theorem 2

The proof follows by observing the moment evolutions of the mass distributions μ_t and using the moment method. The proof is split into several lemmas. Also for notational simplicity we will drop the dependence on α and denote $\mu_\infty(\alpha)$ and q_α by μ and q respectively since α will stay fixed in any argument. Thus

$$\mu := \mu_\infty(\alpha) = \frac{\alpha}{2}\delta(-q) + (1 - \alpha)\Phi_q + \frac{\alpha}{2}\delta(q). \tag{7}$$

Also denote the k^{th} moments of μ_t as $M_k(t)$. We now make some simple observations which are consequences of the previously stated lemmas. Recall the free and frozen mass

distributions ν_t and f_t . We denote the k^{th} moments of the measures ν_t (the free mass at time t), f_t (the frozen mass at time t), by $M_k^\nu(t)$ and $M_k^f(t)$ respectively. Also define \tilde{f}_t and $\tilde{\nu}_t$ similarly to $\tilde{\mu}_t$ in (1). Assuming Conjecture 1, it follows from Lemma 3 that,

$$\tilde{f}_t \xrightarrow{weak} f \quad (8)$$

where $f := \frac{\alpha}{2}\delta(-\ell_\alpha) + \frac{\alpha}{2}\delta(\ell_\alpha)$, and ℓ_α appears in the statement of Conj 1. This implies that

$$\frac{M_k^f(t)}{t^{k/2}} = \begin{cases} 0, & k \text{ odd} \\ \alpha \ell_\alpha^k (1 + o(1)), & k \text{ even} \end{cases} \quad (9)$$

where $o(1)$ goes to 0 as t goes to infinity.

The proof of Theorem 2 is in two steps: first we show that $\ell_\alpha = q$ and then show that $\tilde{\nu}_t$ converges weakly to the part of μ which is absolutely continuous with respect to the Lebesgue measure. Clearly the above two results combined imply Theorem 2.

As mentioned this is done by observing the moment sequence $M_k(t)$. Now notice owing to symmetry of the measures μ_t for any t , $M_{2k+1}(t) = 0$ for all non-negative integers k .

Thus it suffices to consider $M_{2k}(t)$ for some non-negative integer k . We begin by observing that at any time t the change in the moment $M_{2k}(t+1) - M_{2k}(t)$, is caused by the movement of the free mass ν_t . The change caused by a mass m moving at a site x (already argued in the proof of Lemma 1 for $k = 1$) is

$$\frac{m((x+1)^{2k} + (x-1)^{2k})}{2} - mx^{2k} = m \left[\sum_{i=1}^k \binom{2k}{2k-2i} x^{2k-2i} \right]. \quad (10)$$

Now summing over x we get that,

$$M_{2k}(t+1) - M_{2k}(t) = \sum_{i=1}^k \binom{2k}{2k-2i} M_{2k-2i}^\nu(t). \quad (11)$$

Notice that the moments of the free mass distribution ν_t appear on the RHS since m in (10) was the free mass at a site x . Now using (11) we sum $M_{2k}(j+1) - M_{2k}(j)$ over $0 \leq j \leq t-1$ and normalize by t^k to get

$$\frac{M_{2k}(t)}{t^k} = \sum_{j=0}^{t-1} \left[\sum_{i=1}^k \binom{2k}{2k-2i} M_{2k-2i}^\nu(j) \frac{1}{t^k} \right]. \quad (12)$$

Recall that by Lemma 1, for any $k \geq 1$, $M_{2k-2}^\nu(j)$ is $O(j^{k-1})$. Moreover, the above equation allows us to make the following observation:

Claim. Assume (9) holds. Then for any $k \geq 1$, the existence of $\lim_{j \rightarrow \infty} \frac{M_{2k-2}^\nu(j)}{j^{k-1}}$ implies existence of $\lim_{j \rightarrow \infty} \frac{M_{2k}^\nu(j)}{j^k}$.

Proof of claim. Notice that by Lemma 1, $M_{2k-\ell}^\nu(j) = O(j^{k-2})$ for any $\ell \leq 4$. Also let

$$\lim_{j \rightarrow \infty} \frac{M_{2k-2}^\nu(j)}{j^{k-1}} = M_{2k-2}^\nu,$$

which exists by hypothesis.

Thus using (12) and the standard fact that

$$\lim_{t \rightarrow \infty} \sum_{j=0}^{t-1} \frac{j^{k-1}}{t^{k-1}} \frac{1}{t} = \int_0^1 x^{k-1} dx = \frac{1}{k}$$

we get

$$\sum_{j=1}^{t-1} \left[\sum_{i=1}^k \binom{2k}{2k-2i} \frac{M_{2k-2i}^\nu(j)}{t^k} \right] = (2k-1)M_{2k-2}^\nu + o(1) + O\left(\frac{1}{t}\right).$$

Thus

$$\lim_{t \rightarrow \infty} \frac{M_{2k}(t)}{t^k} = (2k-1)M_{2k-2}^\nu \quad (13)$$

and since

$$M_{2k}(t) = M_{2k}^\nu(t) + M_{2k}^f(t), \quad (14)$$

we are done by (9). □

Using the above claim, the fact that $\lim_{t \rightarrow \infty} \frac{M_k(t)}{t^{k/2}}$ and hence, $\lim_{t \rightarrow \infty} \frac{M_k^\nu(t)}{t^{k/2}}$ (by (14) and (9)) exists for all k , follows from the fact that $\frac{M_2(t)}{t} = (1 - \alpha)$ (see (6)). Let us call the limits M_k and M_k^ν respectively.

Thus we have

$$M_{2k} = M_{2k}^\nu + \alpha \ell_\alpha^{2k} = (2k-1)M_{2k-2}^\nu, \quad (15)$$

where the first equality is by (14) and (9) and the second by (13). For $k = 1$ we get

$$\alpha \ell_\alpha^2 + M_2^\nu = 1 - \alpha.$$

Notice that this implies that for all k , M_{2k}^ν can be expressed in terms of a polynomial in ℓ_α of degree $2k$, which we denote as $P_k(\ell_\alpha)$. Then, by (15) the polynomials P_k satisfy the following recurrence relation:

$$\begin{aligned} P_k(\ell_\alpha) &= (2k-1)P_{k-1}(\ell_\alpha) - \alpha \ell_\alpha^{2k} \\ P_0 &= 1 - \alpha. \end{aligned} \quad (16)$$

By definition, we have

$$P_k(\ell_\alpha) = M_{2k}^\nu = \lim_{t \rightarrow \infty} \frac{M_{2k}^\nu(t)}{t^k} = \lim_{t \rightarrow \infty} \frac{\sum_{-\beta_t \leq x \leq \beta_t} x^{2k} \nu_t(x)}{t^k}. \quad (17)$$

Thus assuming Conj. 1 and the fact that $\sum_{-\beta_t \leq x \leq \beta_t} \nu_t(x) = 1 - \alpha$ for all t , we get the following family of inequalities,

$$0 \leq P_k(\ell_\alpha) \leq (1 - \alpha) \ell_\alpha^{2k} \quad \forall k \geq 0. \quad (18)$$

We next show that the above inequalities are true only if $\ell_\alpha = q$ where q appears in (7).

Lemma 5. *The inequalities in (18) are satisfied by the unique number ℓ_α such that*

$$\frac{\alpha}{2}\ell_\alpha = \frac{(1-\alpha)e^{-\ell_\alpha^2/2}}{\sqrt{2\pi}\Phi([- \ell_\alpha, \ell_\alpha])}$$

where $\Phi(\cdot)$ is the standard Gaussian measure.

Thus the above implies that necessarily $\ell_\alpha = q$ where q appears in (7). This was mentioned in Remark 1.

Proof. To prove this, first we write the inequalities in a different form so that the polynomials stabilize. To this goal, let us define

$$\tilde{P}_k = \frac{P_k}{(2k-1)!!}$$

where $(2k-1)!! = (2k-1)(2k-3)\dots 1$. Then it follows from (16) that

$$\tilde{P}_k(\ell_\alpha) = \tilde{P}_{k-1}(\ell_\alpha) - \frac{\alpha}{(2k-1)!!}\ell_\alpha^{2k}.$$

Hence

$$\tilde{P}_k(\ell_\alpha) = \left(1 - \alpha - \sum_{i=1}^k \frac{\alpha \ell_\alpha^{2i}}{(2i-1)!!}\right).$$

The inequalities in (18) translate to

$$0 \leq 1 - \alpha - \sum_{i=1}^k \frac{\alpha \ell_\alpha^{2i}}{(2i-1)!!} \leq \frac{\ell_\alpha^{2k+2}}{(2k+1)!!}. \quad (19)$$

Let us first identify the power series

$$g(x) = \sum_{i=1}^{\infty} \frac{x^{2i-1}}{(2i-1)!!}.$$

Clearly the power series converges absolutely for all values of x . It is also standard to show that one can interchange differentiation and the sum in the expression for $g(\cdot)$. Thus we get that,

$$\frac{dg(x)}{dx} = 1 + xg(x).$$

Solving this differential equation using integrating factor $e^{-x^2/2}$ and the fact that $g(0) = 0$ we get

$$g(x) = e^{x^2/2} \int_0^x e^{-y^2/2} dy.$$

As $k \rightarrow \infty$, the upper bound in (19) converges to 0 for any value of ℓ_α . Also the expression in the middle converges to $1 - \alpha - \alpha \ell_\alpha g(\ell_\alpha)$. Thus taking the limit in (19) as $k \rightarrow \infty$ we get that $\ell_\alpha > 0$ satisfies

$$\ell_\alpha g(\ell_\alpha) = \frac{1-\alpha}{\alpha}. \quad (20)$$

Clearly this is the same as the equation appearing in the statement of the lemma. Also notice that since $xg(x)$ is monotone on the positive real axis, by the uniqueness of the solution of (20) we get $\ell_\alpha = q$ where q appears in (7). Hence we are done. \square

The value of ℓ_α that solves (20) when $\alpha = 1/2$ is approximately 0.878. Figure 2 shows the numerical convergence of β_t/\sqrt{t} to q_α for various values of α .

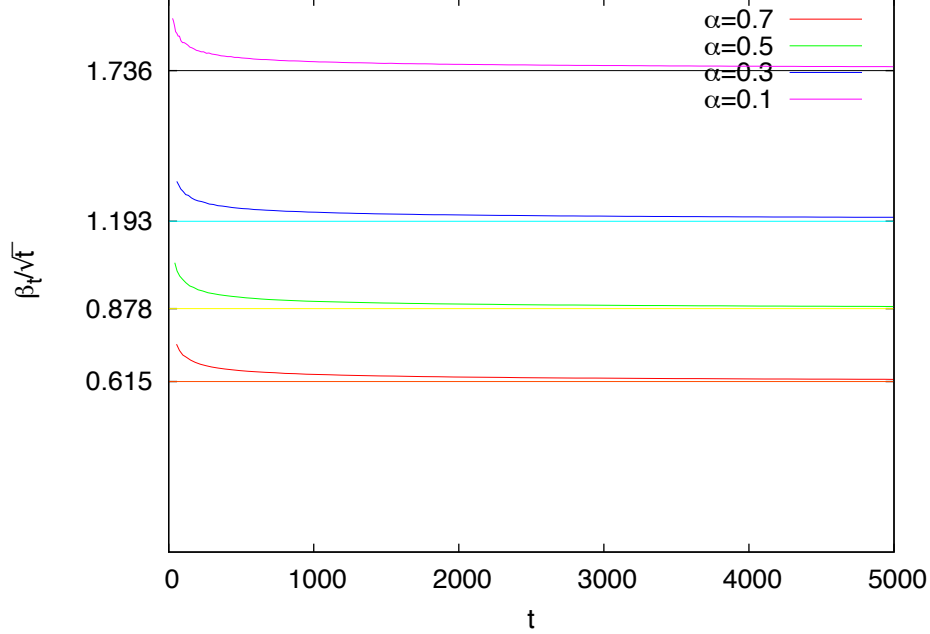


Figure 2: Convergence of β_t/\sqrt{t} for various α . The horizontal lines denote the values q_α and the curves plot $\frac{\beta_t}{\sqrt{t}}$ as a function of time t .

Thus, assuming Conjecture 1, by Lemma 5, \tilde{f}_t converges to f (as stated in (8)) which consists of two atoms of size $\frac{\alpha}{2}$ at q and $-q$. To conclude the proof of Theorem 2, we now show $\tilde{\nu}_t$ converges to the absolutely continuous part of μ (see (7)). Recall that by (17) and Lemma 5 the $2k^{\text{th}}$ moment of $\tilde{\nu}_t$ converges to $P_k(q)$. We will use the following well known result:

Lemma 6 (30.1, [1]). *Let μ be a probability measure on the line having finite moments $\alpha_k = \int_{-\infty}^{\infty} x^k \mu(dx)$ of all orders. If the power series $\sum_k \alpha_k r^k / k!$ has a positive radius of convergence, then μ is the only probability measure with the moments $\alpha_1, \alpha_2, \dots$*

Thus, to complete the proof of Theorem 2 we need to show the following:

Claim. The $2k^{\text{th}}$ moment of the measure $(1-\alpha)\Phi_q$ is $P_k(q)$ where q appears in (7).

To prove this claim, it suffices to show that the moments of $(1-\alpha)\Phi_q$ satisfy the recursion (16). Recall that $q = q_\alpha$. Let $C = C_\alpha := \frac{\sqrt{2\pi}\Phi([-q,q])}{1-\alpha}$. Using integration by parts we have:

$$\begin{aligned} \frac{\int_{-q}^q x^{2k} e^{-\frac{x^2}{2}} dx}{C} &= \frac{\int_{-q}^q x^{2k-1} x e^{-\frac{x^2}{2}} dx}{C} \\ &= -\frac{2q^{2k-1} e^{-\frac{q^2}{2}}}{C} + \frac{(2k-1)}{C} \int_{-q}^q x^{2k-2} e^{-\frac{x^2}{2}} dx. \end{aligned}$$

By the relation that q satisfies in the statement of Theorem 2, the first term on the RHS without the $-$ sign is αq^{2k} . Also, note that the second term is $(2k-1)$ times the $(2k-2)^{nd}$ moment of $(1-\alpha)\Phi_q$. Thus, the moments of $(1-\alpha)\Phi_q$ satisfy the same recursion as in (16).

Now from Example 30.1 in [1], we know that the absolute value of the k^{th} moment of the standard normal distribution is bounded by $k!$. Then, similarly, the absolute value of the k^{th} moment of our truncated Gaussian, Φ_q , is bounded by $c^k k!$ for a constant c . Then Lemma 6 implies that Φ_q is determined by its moments and quoting Theorem 30.2 in [1] we are done. □

3 Concluding Remarks

We conclude with a brief discussion about a possible approach towards proving Conjectures 1 and 2 and some experiments in higher dimensions.

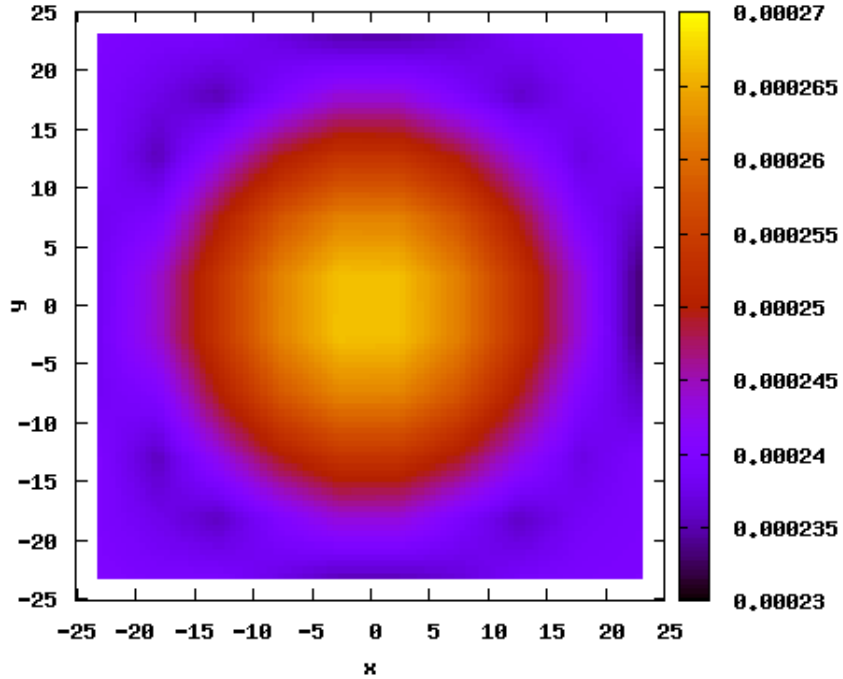


Figure 3: Heat map of the free mass distribution after 1000 steps in 2 dimensions for FBD-1/2.

The free part ν_t of the distribution μ_t could represent the distribution of a random walk in a growing interval. If the interval boundaries grow diffusively, the scaling limit of this

random process will be a reflected Ornstein-Uhlenbeck process on this interval $[-q, q]$. We remark that the stationary measure for Ornstein-Uhlenbeck process reflected on the interval is known to be the same truncated Gaussian which appears in Theorem 2, see [3, (31)]. This connection could be useful in proving the conjectures.

We also note that similar results are expected in higher dimensions; in particular, the mass distribution should exhibit rotational symmetry. See Fig 3. Note the truncated Gaussian shape for the slices $x = 0$ (Fig 4).

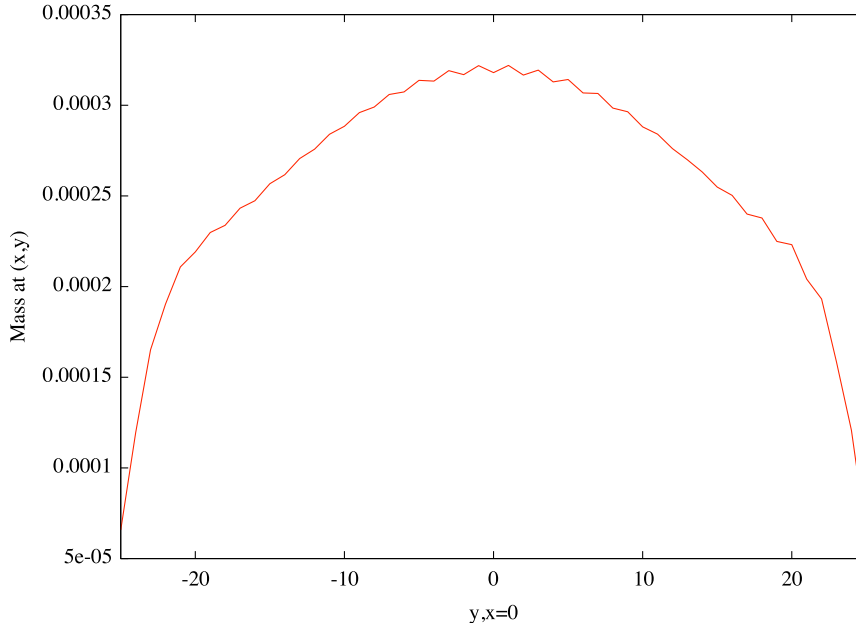


Figure 4: Slice of the free mass distribution at $x = 0$ after 1000 steps in 2 dimensions for the analogue of **FBD**-1/2.

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